Low Risk Sharing with Many Assets*

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February 2024

Abstract

Classical contributions in international macroeconomics reconcile low international risk sharing by generating a non-traded component to exchange rates. However, when there is cross-border trade in just one domestic and one foreign-currency-denominated risk-free asset, such price movements are ruled out by no-arbitrage restrictions. Allowing for within-country heterogeneity in stochastic discount factors, we recover low risk-sharing even with cross-border trade in two risk-free assets, as long as within-country heterogeneity increases when exchange rates depreciate.

*First Draft: November 2023. We thank Sushant Acharya, Nicolas Caramp, Andrés Carvajal, Russell Cooper, Giancarlo Corsetti, Valentin Haddad, Sylvain Leduc, Simon Lloyd, Thomas Mertens, and Alan Taylor, and seminar participants at EUI and UC Davis for helpful comments and suggestions. The views expressed in this article are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Federal Reserve Bank of San Francisco or of anyone else associated with the Federal Reserve System. Remaining errors are our own.

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1. Introduction

From a macroeconomic perspective, exchange rates facilitate the sharing of consumption risk across countries by transforming units of account. From a financial perspective, exchange rates appear in all unhedged cross-border portfolio positions and so are disciplined by a number of arbitrage conditions. These two perspectives can offer contrasting implications for various puzzles in international macroeconomics.

We specifically revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993), which describes the sharing of risk across countries in terms of consumption and relative price co-movements (Obstfeld and Rogoff, 2000). While a large class of macroeconomic models reconciles the cyclicality of exchange rates with the data, these models fail when we account for no-arbitrage restrictions arising from trade in multiple assets. In this paper, we first illustrate why this incongruence between macroeconomic mechanisms and no-arbitrage restrictions arises. We then propose a generalization of these models to allow for within-country heterogeneity in stochastic discount factors (SDFs) and illustrate how this extension reconciles low risk sharing with cross-border trade in many assets.

When international financial markets are complete, a large class of models admits the following relationship:

$$\left(\frac{C_{t+1}}{C^*_{t+1}} / \frac{C_t}{C^*_t}\right)^{\sigma} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where C_t is Home aggregate consumption, C^*_t is Foreign aggregate consumption and $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency. Ex-ante risk-sharing implies ex-post redistribution, which, with trade in risk-free assets, must occur entirely through exchange rate movements. For example, following a positive productivity shock in the Home country, the Home currency depreciates leading to higher consumption abroad. Ex-post exchange rates thus move to reallocate wealth from Home to Foreign and so are "risky" from the perspective of a Home investor. This implication echoes closed economy models with complete markets which imply a correlation of -1 between the representative agent SDF and the market portfolio, see, e.g. Lettau (2002). However, in the data, exchange rates often appreciate when Home consumption rises, constituting the Backus-Smith *puzzle*.

When markets are incomplete, the condition above needs to hold only in expectation and so may fail ex-post. Classical contributions in international macroeconomics such as Corsetti, Dedola and Leduc (2008); Benigno and Thoenissen (2008) show that non-traded risk, which can arise from consumption or production complementarities, enable incomplete market models to generate plausible patterns of international risk sharing. We refer to these economic mechanisms as goods market mechanisms (as opposed to financial).[1] However, these models only allow cross-border trade in a single risk-free asset (denominated in either currency). Lustig and Verdelhan (2019) show that, no-arbitrage restrictions from cross-border trade in at least one Home and one Foreign risk-free asset imply prices always co-move negatively with consumption – so incomplete market models cannot resolve the puzzle of excessive risk-sharing *regardless* of goods market frictions and other economy specifics.[2] Since, in practice, the number of assets traded across borders is very high, this result has far-reaching implications for both theory and practice.

[1]Under the *asset market view of the exchange rates*, the Backus-Smith condition offers a characterization of the financial markets. The seminal contributions of Cole and Obstfeld (1991), Tesar (1993), Stockman and Tesar (1995), Lewis (1996), Fitzgerald (2012) among others, show that goods markets can be just as important in determining the co-movement of international prices and consumption.

[2]See also Benigno and Küçük (2012) in a model of portfolio choice with international trade in two nominal bonds.

Our first contribution is to highlight the mechanism through which goods markets can reconcile the Backus-Smith puzzle. We show that any model that achieves this resolution must generate a non-traded component of relative prices which is "safe" from a domestic investor perspective.[3] Having established this mechanism, we show that moving from cross-border trade in a single risk-free asset to cross-border trade in just one Home and one Foreign risk-free asset inhibits *any* goods-market mechanisms from reconciling the patterns of risk-sharing observed in the data since the additional no-arbitrage restriction rules out such safe exchange rate movements.

We use a common framework, emphasizing macroeconomic fundamentals, to investigate four cases under (i) financial autarky, (ii) financial trade in a single asset, (iii) trade in Home and Foreign currency-denominated risk-free assets, and (iv) trade in risky assets. Under (i), we show that the positive comovement between relative consumption and relative prices can arise from *safe* wealth effects which constitute non-traded risk. Under (ii) where there is cross-border trade in a single (Foreign) risk-free asset, while the Foreign household is insuring against these wealth effects, the Home household may not be sufficiently insured depending on the parameters of the specific macro environment. Our main focus is on case (iii), where there is trade in two *risk-free* assets. Then, households in both countries ex-ante insure these wealth effects – leading to ex-post redistribution and a negative co-movement between relative consumption and prices. In this case, exchange rates themselves are effectively traded – see also Chernov, Haddad and Itskhoki (2023). In case (iv), we show that adding trade in risky assets does not necessarily determine the cyclicality of exchange rates, and goods market mechanisms remain powerful in

[3] The conditions for goods market mechanism that we derive apply not only to models with consumption or production complementarities described above, but also to models with costly consumer search (Bai and Ríos-Rull, 2015), or global value chain integration (Corsetti, D'Aguanno, Dogan, Lloyd and Sajedi, 2023*b*).

resolving the Backus-Smith puzzle.

Our second contribution is to propose generalizations of the representative investor, two-country model to allow the co-existence of multiple SDFs (heterogeneous investors) within the Home country. Domestic heterogeneity implies the presence of non-traded risk within countries which, if correlated with exchange rate movements, is not insured away by additional trade in Home and Foreign risk-free assets. We consider two distinct models which both imply similar conditions under which low risk sharing can be obtained.

We first consider a model with heterogeneous marginal SDFs. There are two investors in the Home economy. Both investors price the domestic currency-denominated risk-free asset while only one investor (of measure zero) prices the Foreign currency risk-free asset. Low risk sharing with many assets can be attained if the marginal investor who participates in Foreign assets is sufficiently exposed to exchange rate movements but does not insure the other investor through domestic asset markets. Domestic market incompleteness is high if the investor participating in Foreign markets earns excess Sharpe ratios or if the covariance of SDFs within the country is low. Using portfolio return data, we discipline the former by ruling out "good deals" (Cochrane and Saá-Requejo, 2000) and we calibrate the covariance using micro data from the literature. Our model can match the facts on international risk sharing with a correlation of non-traded risk and exchange rates of between one-third and one-half.

Our second model considers heterogeneous consumers in the Home economy facing idiosyncratic consumption risk, trading in fully integrated international markets. We build on a large literature investigating whether idiosyncratic risk has aggregate pricing consequences, most closely Constantinides and Duffie (1996), and Krueger and Lustig (2010). We show this consideration carries stark implications for international risk sharing even with trade in multiple assets. The presence of idiosyncratic risk implies an Euler inequality for aggregate consumption, and we show the properties of exchange rates are shaped by the idiosyncratic risk distribution. The Backus-Smith puzzle can be resolved as long as idiosyncratic consumption risk increases sufficiently with exchange rate depreciation. Back of the envelope calculations suggest that empirically plausible heterogeneity can deliver low risk sharing.

Related Literature Most closely related papers to ours are Benigno and Küçük (2012), Lustig and Verdelhan (2019), Chernov, Haddad and Itskhoki (2023), and Jiang, Krishnamurthy and Lustig (2023). Lustig and Verdelhan (2019) consider multiplicative incomplete market wedges as in Backus, Foresi and Telmer (2001). Benigno and Küçük (2012) and Lustig and Verdelhan (2019) show that introducing a second internationally traded bond breaks the ability of international macro models to reconcile the Backus-Smith puzzle. We extend their frameworks beyond the representative agent assumption, generalizing some of their results, and show how within-country heterogeneity in SDFs can generate low risk sharing as we increase the number of internationally traded assets.

Chernov et al. (2023) investigate how different financial market structures and the mix of locally, globally traded, and unspanned risks contribute to different exchange rate puzzles. Models with heterogeneous marginal investors naturally relate to models of intermediation (Gabaix and Maggiori (2015); Itskhoki and Mukhin (2021)), but our framework can specifically be viewed as a way of breaking global risks into local risks.

Using a wedge accounting framework, Itskhoki and Mukhin (2023) find that financial shocks can reconcile exchange rate puzzles. Jiang, Krishnamurthy and Lustig (2023) show that Euler equation wedges are necessary to resolve Backus-Smith puzzle. We complement their analysis by structuring these wedges and allowing for multiple within-country SDFs connected by risk-sharing.[4]

From a finance perspective, Bakshi, Cerrato and Crosby (2018) also allow for multiple SDFs considering additive wedges, but their focus is on isolating the spanned and unspanned components and generalizing the results in Brandt, Cochrane and Santa-Clara (2006). Sandulescu, Trojani and Vedolin (2021) extract minimum variance and minimum entropy SDFs and show that the Backus-Smith condition holds with their model-free SDFs. Orłowski, Tahbaz-Salehi, Trojani and Vedolin (2023) extend the result of Lustig and Verdelhan (2019) to allow for varying degrees of financial integration and different market structures with no-arbitrage pricing.

Our work is also related to the broader literature on market segmentation in international macro. Maggiori, Neiman and Schreger (2020) provide evidence of segmentation in international fixed-income markets. Christelis, Georgarakos and Haliassos (2013) explore the determinants of portfolio differences across countries. Cociuba and Ramanarayanan (2019) build a model of endogenously incomplete domestic markets using the framework of Alvarez, Atkeson and Kehoe (2002) and show that the Backus-Smith condition need only hold for households active in international financial markets. Kollmann (2012) and Chien, Lustig and Naknoi (2020) also model heterogeneous participation and the latter focus on equity market participation. A key difference is that these models assume complete markets internationally. We build on their contributions and study the general case of incomplete international markets.

When we turn to a model with heterogeneous consumers, we tie to a large literature investigating whether idiosyncratic risk affects aggregate Euler, see, e.g. Mankiw (1986); Weil

[4] Colacito and Croce (2013) and Farhi and Gabaix (2016) investigate the role of long-run risk and rare disasters respectively in generating observed correlations between cross country asset returns and exchange rate returns. For a treatment of optimal portfolio choice and international risk sharing, see Devereux and Sutherland (2011) and Heathcote and Perri (2013) amongst others.

(1992); Constantinides and Duffie (1996); Krueger and Lustig (2010); Werning (2015); Kaplan, Moll and Violante (2018). Acharya and Pesenti (2024) investigate the role of precautionary savings in generating monetary policy spillovers in a two-country open economy model.[5] We take a consumption asset pricing approach and assess the relevance of consumption heterogeneity for determining exchange rates.

The rest of the paper is organised as follows. Section 2 provides a characterization of how goods markets drive exchange rate cyclicality and how this mechanism works with trade in risk-free and/or risky assets. Section 3 proposes our generalization of incomplete markets with SDF heterogeneity and derives the minimum bounds necessary for generating empirical risk-sharing patterns with different financial structures. Section 4 concludes.

2. Two-country, representative agent, incomplete markets

Consider a two-country model where M_{t+1} denotes the Home representative household's SDF and M^*_{t+1} denotes the Foreign representative household's SDF. Home and Foreign households each trade their respective domestic risk-free real bonds with returns R_t and R^*_t respectively frictionlessly (i.e. no borrowing constraints). No-arbitrage pricing implies:[6]

[5] In the open economy macro literature, for papers on heterogeneous SDFs arising out of consumption or preference heterogeneity see Ramchand (1999); Ghironi (2006); Farhi and Werning (2016); Fornaro (2018); De Ferra, Mitman and Romei (2020); Hong (2020); Auclert, Rognlie, Souchier and Straub (2021); Kekre and Lenel (2021); Cugat (2022); Zhou (2022); Guo, Ottonello and Perez (2023); Chen, Devereux, Shi and Xu (2023); Acharya and Challe (2024).

[6] E.g. in the case of time-separable CRRA utility $M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-s}$. Alternatively, in the tradition of Hansen and Jagannathan (1991), the SDFs are simply a risk-return operator implied by the traded assets. See Section 4.

$$\mathbb{E}_t[M_{t+1}] = 1/R_{t+1}, \tag{2}$$

$$\mathbb{E}_t[M^*_{t+1}] = 1/R^*_{t+1} \tag{3}$$

If the Home (Foreign) households also trade the Foreign (home) bond, and the real exchange rate at time t is denoted with $\mathcal{E}_t$, then we obtain the following two Euler conditions:

$$\mathbb{E}_t\left[M^*_{t+1}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] = 1/R_{t+1}, \tag{4}$$

$$\mathbb{E}_t\left[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1/R^*_{t+1}. \tag{5}$$

We assume SDFs, allocations and prices are jointly log-normal.[7] To close the model without explicitly specifying goods markets, an exchange rate process is needed, which is consistent with equations (2)–(5) above. This problem reduces to finding an exchange rate process that satisfies:[8]

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \tag{6}$$

where $x = log(X)$. Naturally, the process corresponding to complete markets ($\Delta e_{t+1} = m^*_{t+1} - m_{t+1}$) is one candidate. More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (6):

$$\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1} \tag{7}$$

where η_{t+1} is an incomplete markets wedge which must satisfy certain conditions imposed by asset trade.[9] The Backus-Smith condition (1) restricts the covariance between relative SDFs

[7] Our results generalize to non log-normal settings using entropy expansions (Lustig and Verdelhan, 2019).

[8] See Appendix A.1 for full derivation

[9] For a closed form solution of the incomplete markets wedge in a two-country open economy model, see Pavlova and Rigobon (2007).

and exchange rate growth, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, to be positive. We refer to this covariance term as the cyclicality of exchange rates.

Combining (2) and (4), with (7) – which implies that the Home bond is internationally traded – yields:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{8}$$

Combining (3) and (5), with (7) – which implies that the Foreign bond is internationally traded – yields:

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m^*_{t+1}, -\eta_{t+1}) \tag{9}$$

These two conditions mirror those studied by Lustig and Verdelhan (2019) and bound the joint dynamics of the incomplete market wedge and the SDFs, carrying strong implications for the macro side of the model. The conditions reflect the return, for a Home and a Foreign investor respectively, from the non-traded component of exchange rates which must compensate for both volatility of returns and the riskiness of the exchange rate.

While the results we derive are preference-free, to relate back to international macro models, consider the case where agents have time-separable, CRRA preferences over consumption. Then, the incomplete markets wedge is related to:

$$\eta_{t+1} = log \underbrace{\left(\frac{P_{t+1}}{P_t} \frac{P^*_t}{P^*_{t+1}} \right)}_{\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}} - log \underbrace{\left(\frac{C_t}{C_{t+1}} \frac{C^*_{t+1}}{C^*_t} \right)^s}_{\frac{M_{t+1}}{M^*_{t+1}}}$$

where P_t is the Home price level, C_t is aggregate consumption, s is the CRRA coefficient, and terms with asterisks denote the corresponding Foreign objects. The wedge, η, is often interpreted as the non-traded component of exchange rate movements or the wealth gap, see e.g. Corsetti, Dedola and Leduc (2023*a*).

2.1. International Risk-Sharing with Trade in Risk-free Assets

Having now specified our framework, we illustrate the mechanism through which goods market frictions in incomplete market models can help reconcile the pattern of international risk-sharing.

Proposition 1 (One Int'l Traded Asset, Representative Agent No-Arbitrage).

When only Foreign bonds are internationally traded such that equations (2)*,* (3) *and* (5) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if*

$$cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq var_t(m^*_{t+1} - m_{t+1}) \quad (10)$$

where,

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) + var_t(m_{t+1}) \quad (11)$$

Proof. See Appendix A.2. □

The RHS of condition (10) is equal to the volatility of the exchange rate growth under complete markets and is strictly positive. The condition is satisfied if either the non-traded component η_{t+1} leads to relative price fluctuations which are ex-post safe from the perspective of a Home investor, as captured by $cov_t(m_{t+1}, \eta_{t+1}) > 0$ or that the volatility of the non-traded component is high.[10] Equation (11) shows that non-traded risk results in relative price fluctuations which are particularly *safe* when the Home SDF is very volatile or when international comovement in SDFs is low relative to the comovement of exchange rates and the Home SDF.[11] In Section 2.4, using a two-country model with multiple goods, we investigate

[10] Note, that this is consistent with the idea that exchange rate movements exacerbate consumption risk. A low level of consumption implies a high discount factor, and is associated with a depreciation.

[11] Brandt et al. (2006) show that SDFs must comove very strongly to explain the relatively low exchange rate volatility in the data.

the parametric restrictions consistent with this condition.

This conditions provides a general characterization for goods market mechanisms developed to resolve the Backus-Smith puzzle in models with consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008) as well as in models with border effect (Devereux and Hnatkovska, 2011), costly consumer search (Bai and Ríos-Rull, 2015), or global value chain fragmentation (Corsetti et al., 2023*b*), amongst others.

A limitation of Proposition 1, and the models which satisfy it, is that it may exacerbate other exchange rate puzzles– in particular, that of excess volatility of exchange rates. The RHS of equation (10) is equal to the volatility of exchange rates under complete markets, and models with a low volatility will generally fare better in resolving the cyclicality puzzle.

Corollary 1 (Two Int'l Traded Asset, Representative Agent No-Arbitrage).

As the Home bond also becomes internationally traded without arbitrage, $cov_t(m_{t+1}, \eta_{t+1}) \rightarrow$ -$\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2} var_t(\eta_{t+1})$ *as in equation* (8)*, then condition* (10) *implies* $var_t(\Delta e_{t+1}) \leq 0$*.*

Proof. See Appendix A.2. □

Cross-border trade in a second risk-free asset prevents the model from reconciling the Backus-Smith puzzle because this constrains the ex-post *safety* of relative price movements due to non-traded risk. From Equation (8), when the non-traded component becomes safe for Home investors, lower returns on their Foreign bond holdings ($\mathbb{E}_t[\eta_{t+1}]$) translate to higher returns on Home bonds for Foreign investors ($\mathbb{E}_t[-\eta_{t+1}]$). This is only consistent with no arbitrage if the non-traded component becomes sufficiently risky for Foreign investors which would restore the Backus-Smith correlation.

2.2. International Risk Sharing with Trade in Risky Assets

If instead of allowing for trade in both Home and Foreign risk-free assets, we allow for trade in Home and Foreign risky assets, then trade in assets does not necessarily restrict the cyclicality of exchange rates.[12] In practice, few assets traded across borders are risk-free in real terms, so this case is likely to be a better approximation of reality. Risky assets could include equity or long maturity bonds. In this case, equations (2)–(5) are replaced by:

$$\mathbb{E}_t[M_{t+1}\tilde{R}_{t+1}] = 1, \tag{12}$$

$$\mathbb{E}_t[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\tilde{R}^*_{t+1}] = 1, \tag{13}$$

$$\mathbb{E}_t[M^*_{t+1}\tilde{R}^*_{t+1}] = 1, \tag{14}$$

$$\mathbb{E}_t[M^*_{t+1}\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{-1}\tilde{R}_{t+1}] = 1, \tag{15}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Proposition 2 (Risky Assets, Representative Agent No-Arbitrage).

When only risky Home and Foreign assets are internationally traded such that equations (12) *-* (15) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if*

$$var_t(\Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \leq 0 \tag{16}$$

Proof: See Appendix A.3. □

Trade in risky assets is not sufficient to determine the cyclicality of exchange rates, unless the risky returns are uncorrelated with domestic non-traded risk. Adding cross-border trade in the Home risk free bond will reimpose equation (8) and cross-border trade in the Foreign risk

[12] Lustig and Verdelhan (2019) derive the restrictions imposed by trade in risky assets in addition to two-risk free bonds. In their environment, trade in risky assets can therefore not break patterns of risk sharing.

free bond will reimpose equation (9). Consider the environment in Proposition 1, where only Foreign risk-free bonds are internationally traded. Introducing trade in a Home risky assets does not necessarily recover the strong risk-sharing implications that arise when international trade in a second risk-free asset is allowed.

Corollary 2

When Foreign risk-free bonds are internationally traded such that equations (2), (3) *and* (5) *hold, as well as a Home risky asset is internationally traded such that equations* (12) *and* (15) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if*

$$var_t(\Delta e_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}_{t+1}) \leq 0 \tag{17}$$

Proof. *Additionally* imposing equation (9) implies $cov_t(\eta_{t+1}, \tilde{r}^*_{t+1}) = 0$. The result to Corollary 2 then follows from Proposition 2. See Appendix A.3 for additional steps. □

Corollary 2 also shows that as enough assets become traded, so that we approach complete markets, $\sigma_\eta \to 0$, recovering the impossibility result of Corollary 1.

2.3. Degree of Market Completeness and Risk-Sharing

To illustrate the importance of this puzzle, we turn to a framework in the tradition of Lucas (1978), where agents have time-separable, CRRA preferences over an exogenous consumption stream. In particular:

$$\Delta c_{t+1} = \sum_{k=1}^{N} g_{y_{k,t+1}}, \tag{18}$$

$$m_{t+1} = -s\Delta c_{t+1}, \tag{19}$$

where $g_{y_{k,t+1}} = y_{k,t+1} - y_{k,t} \sim i.i.d\ \mathcal{N}(\mu_{y_k}, \sigma_{y_k})$ denotes the growth rate of k-th productive unit that comprises the consumption good. Corresponding variables for the Foreign economy are denoted with an asterisk. Start with the case of $N = 1$ productive units, discussed in (Lustig and Verdelhan, 2019, Sec III.A). Frictionless international trade in Home and Foreign risk-free bonds (Equations 8 & 9) and additional trade in a Home and a Foreign risky asset (a claim on $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$ respectively) will imply that the incomplete markets wedge η_{t+1} is orthogonal to $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$, and it then follows that the only equilibrium is $\eta_{t+1} = 0$– i.e. markets are complete. When $N > 1$, additional risky claims need to traded to complete the market. However, for any N, frictionless international trade in just the Home and the Foreign real bonds ensures risk-sharing consistent with complete markets (Corollary 1). Relatedly, Chernov et al. (2023) show that incompleteness alone cannot match exchange rate puzzles, unless there is some lack of integration in financial markets (i.e. only Foreign bonds traded as in Corollary 2).

Next, we generalize this framework to a two-good economy at the autarky limit, to illustrate the macroeconomic forces underpinning the non-traded component η_{t+1}.

2.4. An Equilibrium Model in the Autarky Limit

To relate as closely as possible to the prevailing resolutions of the Backus-Smith puzzle in international macroeconomics literature, we specify a model capturing key ingredients in the literature and use it as a basis for constructing the investor SDFs.

The representative agent derives per-period utility from consumption:

$$u(C_t) = \beta \frac{C_t^{1-s}}{1-s} \tag{20}$$

where β is the discount factor,[13] and the consumption bundle is given by:

$$C_t = \left[\alpha^{\frac{1}{\phi}} C_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_{F,t}^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}} \tag{21}$$

where ϕ is the trade elasticity and α is the measure of home-bias. The domestic budget constraint can be written as :

$$C_t + \frac{P_{H,t}}{P_t} Y_{H,t} \leq R_t B_{t-1} - B_t + \mathcal{E}_t R_t^* B_t^* - \mathcal{E}_t B_{t-1}^* \tag{22}$$

and we rewrite $\frac{P_{H,t}}{P_t} Y_{H,t} = Y_t$. Foreign agents face an analogous maximization. However, Foreign agents only trade in the Foreign denominated bond. Goods market clearing requires:

$$C_{H,t} + C_{H,t}^* = Y_{H,t}; \quad C_{F,t} + C_{F,t}^* = Y_{F,t}^*$$

where endowment process are given by $Y_{H,t} = \rho Y_{H,t-1} + (1-\rho) Y_H + \epsilon_t$, $Y_{F,t}^* = \rho Y_{F,t-1}^* + (1-\rho) Y_F^* + \epsilon_t^*$ and ϵ_t and ϵ_t^* are iid $N(0, \sigma_\epsilon)$.

The corresponding price level is given by:

$$P_t = \left[\alpha P_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha) P_{F,t}^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}} \tag{23}$$

and P^* is defined symmetrically. The real exchange rate is given by $\mathcal{E} = P^*/P$. We relegate a full description of the model to Appendix A.4.

Lemma 1 describes the autarky limit for prices and allocations, attained at the limit of zero liquidity (Corsetti et al., 2008) and full home-bias limit (Itskhoki and Mukhin, 2021, 2023).

Lemma 1 (Autarky Limit).

In the autarky limit $\alpha \to 1$, $B, B^* \to 0$, the model is summarized by the following equations

[13]The discount factor can be used as a stationarity-inducing device, see Appendix B.

$$m_{t+1} = -sg_{y_{H,t+1}},$$

$$m^*_{t+1} = -sg_{y_{F,t+1}},$$

$$\Delta e_{t+1} = \frac{1}{1-2(1-\phi)}(g_{y_{H,t+1}} - g_{y_{F,t+1}}),$$

$$\eta_{t+1} = (g_{y_{H,t+1}} - g_{y_{F,t+1}})\frac{1-s}{1-2(1-\phi)}$$

where $g_{y_{t+1}} = y_{t+1} - y_t$. *It follows that if* $Y_{H,t}$, $Y_{F,t}$ *are log-normally distributed, then* $m_{t+1}, m^*_{t+1}, \eta_{t+1}$ *and* Δe_{t+1} *are jointly normally distributed.*

Our approximation technique relies on taking the autarky limit for real quantities. There are two reasons why this limit is attractive. First, we prove the joint log normality of SDFs and the exchange rate at the full home-bias limit as $\alpha \to 1$, whereas this is not the case for $\alpha < 1$. Second, the allocations and prices are invariant to the number of assets traded.[14] Then, whereas we calculate the covariance in the financial autarky case by direct computation, we back out the implied covariance at the autarky limit, denoted $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, between SDFs and depreciations when there is trade in assets by imposing $cov_t(m^{(*)}_{t+1}, \eta_{t+1})$ consistent with (8)-(9).

Proposition 3 (Representative agent Backus-Smith at the autarky limit).

The two-country model at the autarky limit can deliver $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *conditional on shocks to* $y_{H,t}$ *in the following cases*

i. *when no assets traded*:

$$\frac{-s}{1-2(1-\phi)} \leq 0 \tag{24}$$

[14]While the invariance and the log-normality properties also hold at the complete markets allocation or Cole and Obstfeld (1991) limit, these cannot by construction break the Backus-Smith condition.

ii. *with trade in one risk-free asset*:

$$\frac{-s(1-s)}{1-2(1-\phi)} \geq s^2 - \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 \tag{25}$$

iii. with two assets need $\phi \to \infty$ such that $var_t(\Delta e_{t+1}) \to 0$.

Proof. See Appendix A.2. □

As in Corsetti et al. (2008), allowing for a sufficiently low trade elasticity implies that following an increase in Home productivity, demand for Home goods rises so much, that prices must adjust to constrain Foreign consumption of the Home good for markets to clear. A further interesting point is that under autarky the ability of the model to match risk sharing depends only on ϕ given that $s > 0$.

When there is trade in assets, condition (10) additionally depends on the inter-temporal elasticity of substitution, s. As a result it is not necessarily true that it is harder for the model to replicate Backus-Smith when there is trade in assets. Figure 1 in Appendix A.4 shows the range of values for which condition (25) is satisfied. Looking at the case (ii), the first term in the RHS is the complete markets exchange rate process. As $s \to 0$, this quantity goes to zero. At the same time, the second term on the RHS which is the volatility of non-traded component $\sigma^2_{\eta_{t+1}}$ rises. Therefore, the inequality is satisfied for any value of trade elasticity ϕ. In contrast, with trade in two risk-free bonds, case (iii), a zero covariance between relative SDFs and the exchange rate arises in the limit where $var_t(\Delta e_{t+1})$ approaches zero, but a negative covariance can never be achieved.

Our results do not rely on log-normality per se. Away from this limit, our results can be derived using entropy expansions as in Lustig and Verdelhan (2019) following Backus, Chernov and Zin (2014) amongst others. The challenge with entropy expansions is that the distribution cannot easily be described in closed form, and therefore we cannot tie these result back to macroeconomic quantities.

In Appendix B, we use a calibrated two-country open economy model of Corsetti et al. (2008) to capture the mechanisms away from the financial autarky limit and show that the results derived here continue to hold. In particular, Figure 4 illustrates that the volatility of non-traded component rises by an order of magnitude relative to other components, so the inequality (10) is satisfied and Proposition 1 continues to hold in their baseline calibration.

3. Models with heterogeneous SDFs

We now show that models with heterogeneous investors can reconcile the Backus-Smith anomaly even when two risk-free assets are internationally traded. We consider two distinct models. The first model features two SDFs – one SDF corresponds to a marginal investor who prices only the Home risk-free bond, and the second SDF is for a marginal investor who prices both the Home and the Foreign risk-free bonds. The second model considers a continuum of SDFs, where all investors can frictionlessly participate in both bond markets (i.e. no borrowing constraints) but face uninsurable idiosyncratic consumption risk.

3.1. A model with limited asset market participation.

Consider now the case where Home financial markets are incomplete. The Foreign economy has a representative investor who can frictionlessly buy Home and Foreign risk-free bonds. The Home economy has two investors characterized by SDFs, M and $\hat{M}$. They both participate frictionlessly in the Home risk-free bond market, but only one of the two Home investors participates in the Foreign risk-free bond market. We assume that the investors who participate in Foreign risk-free bonds are measure zero. This model is characterized by equations (2), (3), (4) and

$$\mathbb{E}_t \left[\hat{M}_{t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right] = 1/R^*_{t+1}, \tag{26}$$

where we define:

$$\hat{M}_{t+1} = M_{t+1} D_{t+1} \tag{27}$$

where D_{t+1} captures the degree of heterogeneity in the Home country and $D_{t+1} \neq 1$ for at least some t. This setting can capture a variety of models: $\hat{M}_{t+1}$ may be the intermediaries' SDF in a model akin to Gabaix and Maggiori (2015).

We assume that only the exchange rate markets are segmented within the Home economy, but we allow all Home investors to frictionlessly trade a Home risk-free bond.[15] Therefore, their marginal utility growth will be equated in expectation:

$$\mathbb{E}_t[M_{t+1}] = \mathbb{E}_t[\hat{M}_{t+1}] \tag{28}$$

Since $\hat{M}_{t+1}$ prices both Home and Foreign bonds, $\hat{M}_{t+1}$ satisfies all conditions in Lustig and Verdelhan (2019) and will comove with exchange rates according to the following analogue of equation (1):

$$\mathbb{E}_t \left[\hat{M}_{t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right] = \mathbb{E}_t \left[M^*_{t+1} \right] \tag{29}$$

From equation (28), we derive the following condition on the heterogeneity:

$$\mathbb{E}_t[d_{t+1}] + \frac{1}{2} var_t(d_{t+1}) + cov_t(m_{t+1}, d_{t+1}) = 0 \tag{30}$$

[15] From Proposition 2 and Corollary 2, we can generalize our setup to allowing domestic investors to participate in domestic risky asset markets frictionlessly. The main restriction we require is there be domestic segmentation constraining participation in Foreign risk-free asset markets for a large enough measure of Home investors.

Critically, equation (30) implies that d_{t+1} cannot be an asset specific discounting factor for the marginal investor – i.e. a convenience yield on specific bonds. Heterogeneity is therefore strictly on the investor, as opposed to the asset side. Additionally, d_{t+1} is non-traded risk, since by equations (27) and (28) it follows that d_{t+1} does not affect domestic asset prices. Allowing agents to additionally trade in risky assets further restricts heterogeneity, as expected. In particular, building on Corollary 2, we can show if m and $\hat{m}$ trade in $\tilde{r}$, then $cov(d, \tilde{r}) = 0$.[16]

The extended model admits the same process for exchange rates but a different set of equilibrium restrictions apply to the wedge η_{t+1}.[17] Specifically, equation (8) is unchanged because the Home bond continues to be traded frictionlessly across markets, but market segmentation with respect to the Foreign bond implies the equation (9) is replaced by:

$$\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m^*_{t+1}, d_{t+1}) + \mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1})$$
$$\cdots + cov_t(m^*_{t+1} + d_{t+1}, \eta_{t+1}) = 0 \tag{31}$$

We next derive restrictions on the dynamics of investor heterogeneity and exchange rates, required to match the patterns of international risk sharing observed in the data.

Proposition 4 (Heterogeneous Marginal Investors).

The two-country model with two internationally traded bonds and heterogeneous marginal investors in the Home country can deliver $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if

$$1 \geq \rho_{d_{t+1}, -\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(d_{t+1})} \tag{32}$$

where $\rho_{d_{t+1}, -\Delta e_{t+1}} \equiv \dfrac{cov_t(d_{t+1}, -\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(d_{t+1})}$.

[16] By analogy to Corollary 2, if domestic agents trade in a complete set of securities, $\sigma_d \to 0$.

[17] Note that potentially an exchange rate process with η_{t+1} replace by $d_{t+1} + \tilde{\eta}_{t+1}$ could be used. Instead of making this assumption, we just allow for different restrictions to apply on η_{t+1}. Our results would be unchanged.

Proof: See Appendix A.2. □

The inequality in Proposition 4 describes the joint dynamics of exchange rates and domestic market incompleteness required to break the covariance between SDFs and exchange rates implied by equation (1), when there is trade in both Home and Foreign bonds. First, a necessary condition is that $\frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(d_{t+1})} < 1$– i.e. there is sufficient domestic market incompleteness relative to the volatility of exchange rates. Since this condition is a critical component for our theory, we evaluate its empirical plausibility in Section 3.1.1.

Proposition 4 also bounds the sign of the correlation of SDF heterogeneity (non-traded risk) and exchange rate appreciation to be positive– as should be expected in theory. Consider the Backus-Smith condition (1) where the Home SDF is replaced by $\hat{M}_{t+1}$. Periods of depreciation $\mathcal{E}_{t+1} > \mathcal{E}_t$ are associated with $\hat{M}_{t+1}$ falling (relatively high $\hat{C}_{t+1}$ is associated with low P_{t+1} due to risk sharing). For relatively stable M_{t+1}, D_{t+1} must fall– signifying C_{t+1} is low relative to $\hat{C}_{t+1}$, ceteris paribus. The sufficient condition is therefore that the marginal Home investor in Foreign bond does not provide enough insurance to the Home household against exchange rate movements through the Home asset markets. We provide an example in Appendix C.

Corollary 3

As $\sigma_t(d_{t+1}) \to 0$*, the model collapses to a representative agent economy and condition* (32) *is violated.*

Intuitively, heterogeneous marginal investors allow the model with international trade in two risk-free assets to reproduce the Backus-Smith anomaly as long as the volatility of the difference in Home SDFs is sufficiently high, and the covariance between their SDF differences and the exchange rate is sufficiently positive.

3.1.1 How much heterogeneity?

We now make a first pass at evaluating the plausibility of the conditions under which our 2 SDF framework can reproduce a correct pattern of risk-sharing. We begin by estimating $\hat{M}_{t+1}$ and M_{t+1} in the spirit of Hansen and Jagannathan (1991).

Measuring M and $\hat{M}$ Consider a Home investor trading in equities:

$$\mathbb{E}_t[M_{t+1}R^e_{t+1}] = 1 \tag{33}$$

where R^e_{t+1} is the return on equity. Then, we use the Hansen and Jagannathan (1991) bounds to back out a measure for $var_t(m_{t+1})$.[18] We do not measure $var_t(\hat{m}_{t+1})$ directly. Rather, we leverage the concept of "Good-Deal Bounds" of Cochrane and Saá-Requejo (2000), and ask: what additional Sharpe ratio can the domestic investor earn, by participating in the Foreign markets (like $\hat{m}$)?

Lemma 2 (Limits on heterogeneity and no good deals).

We consider equilibria where we rule out good deals where the Sharpe ratio is $K \geq 1$ times the maximal domestic Sharpe Ratio. Then:

$$(K-1)var_t(m_{t+1}) \geq -2\mathbb{E}_t[d_{t+1}]$$

Proof. See Appendix A.2. □

[18]Assuming $\mathbb{E}_t[M_{t+1}] = 1$ and rearranging yields:

$$var_t(M_{t+1}) \geq sup \left(\underbrace{\frac{\mathbb{E}_t[R^e_{t+1}] - R_{t+1}}{\sqrt{var_t(R^e_{t+1})}}}_{\mathbb{E}_t[SR_{t+1}]} \right)^2 \tag{34}$$

The right hand side (RHS) of the condition above is the squared Sharpe ratio. To maximize the RHS, we choose a high return to variance domestically-traded asset such as equity.

The case of $K \leq 1$ corresponds to a world where the maximal Sharpe ratio available to the investor who can access Foreign markets ($\hat{m}_{t+1}$) is no higher than that of the domestic asset investor (m_{t+1}). The, risk-sharing within Home economy implies: $var_t(m_{t+1}) = var_t(\hat{m}_{t+1})$.

Measuring d We now measure the amount of heterogeneity and incompleteness in the domestic economy. Specifically, we look for a plausible values for $\sigma_{d_{t+1}}$. A sufficiently high value makes it more plausible that our generalized model resolves the Backus-Smith puzzle even when there is trade in two risk-free assets.

Lemma 3 (Domestic market incompleteness and no good deals).

Assume now that there are no good-deals, such that $var_t(\hat{m}_{t+1}) \leq K var_t(m_{t+1})$. *Then:*

$$var_t(d_{t+1}) \leq var(m_{t+1}) \left[1 + K(1 - \frac{2}{\sqrt{K}} \rho_t(\hat{m}_{t+1}, m_{t+1})) \right] \tag{35}$$

Proof. See Appendix A.2. □

Finally, we evaluate the above expression. First, we use standard values from the literature. We take a Sharpe ratio of 0.5 annually implying $var(m_{t+1}) = 0.5$ as in Lustig and Verdelhan (2019). This is on the conservative side, since the gross Sharpe ratio on the S&P 500 is just above 0.6 from time-series momentum strategies (Babu, Levine, Ooi, Pedersen and Stamelos, 2020). Secondly, $\rho_t(\hat{m}_{t+1}, m_{t+1})$ is the correlation between the two SDFs of domestic investors. Zhang (2021) measures correlations between SDFs of various agents (domestic and foreign). They find a value of 0.5 for the correlation between the domestic and the stockholder SDFs, and a value of 0.21 for within country correlation between the stockholders' and the non stockholders' SDFs. A lower correlation would provide a better fit for our model as can be seen from Lemma 4. So in order to be conservative, we set the the correlation between the two

SDFs of domestic investors to the higher value of 0.5.

For deriving the no good-deal bounds, we first use $K = 1$, ruling out the possibility that there are high Sharpe ratios to be had in markets. In this case,

$$var_t(d_{t+1}) = var_t(m_{t+1}) = 0.5$$

From Proposition 4, what matters then is the ratio $\frac{\sigma_{\Delta e_{t+1}}}{\sigma_{d_{t+1}}}$. In the data, $var_t(\Delta e_{t+1}) = 0.11$, see e.g. Lustig and Verdelhan (2019), Lloyd and Marin (2023). As a result, for reconciling the Backus-Smith anomaly, our model requires that the correlation of heterogeneity with exchange rate growth be sufficiently low, where the threshold is given by

$$\rho^{K=1}_{d_{t+1},\Delta e_{t+1}} \leq -\frac{0.33}{0.7} \tag{36}$$

Next, we evaluate a more empirically realistic scenario. We leverage the finding in Barroso and Santa-Clara (2015) that carry trade exposure can double the Sharpe ratio of a diversified stock-bond portfolio, i.e. $K \leq 2$.[19] This would imply $var_t(d_{t+1}) = 0.89$ and therefore the threshold correlation between exchange rate growth and SDF heterogeneity is now:

$$\rho^{K=2}_{d_{t+1},\Delta e_{t+1}} \leq -\frac{0.33}{0.89} \tag{37}$$

3.2. A model with heterogeneous consumers

The starting point of our analysis is the consumption asset pricing model (CAPM) in Rubinstein (1974); Lucas (1978); Breeden (1979), and the two-country model in Lucas (1982), as formulated in Lustig and Verdelhan (2019). We propose a parsimonious extension of this model to allow

[19] There is substantial variation in the maximum annualized Sharpe ratio documented in the literature. Jordà and Taylor (2012), Asness, Moskowitz and Pedersen (2013), and Burnside, Cerrato and Zhang (2020) find strategies with Sharpe ratio as high as 2.42, 1.59, and 3.73 respectively. Lustig, Roussanov and Verdelhan (2011), Menkhoff, Sarno, Schmeling and Schrimpf (2012), and Hassan and Mano (2019) find currency trade strategies with Sharpe ratio of 0.99, 0.95, and 0.69 respectively.

for heterogeneous SDFs within countries in the spirit of Constantinides and Duffie (1996).

We restrict attention to a stochastic discount factor based on a time-separable constant relative risk aversion utility function:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-s} \tag{38}$$

where β is the discount factor, s is the inverse of the intertemporal elasticity of substitution, and consumption growth $\frac{C_{t+1}}{C_t} \equiv \exp\left(\Delta c_{t+1}\right)$ is a random variable drawn from a lognormal distribution which implies:

$$\Delta c_{t+1}^{(*)} = w_{t+1}^{(*)} \sim \mathcal{N}(\mu_{C_t^{(*)}}, \sigma^2_{C_t^{(*)}})$$

where $\mu_{C_t^{(*)}}$ and $\sigma^2_{C_t^{(*)}}$ are conditional mean and variance of this process, and the asterisks denote the corresponding setup for the Foreign economy.

We extend this representative agent environment to allow for a continuum of consumers indexed by i in the Home economy. Each of these consumers faces uninsurable idiosyncratic consumption risk and trades frictionlessly in Home and Foreign risk-free bonds. Critically, building on a large literature assessing whether idiosyncratic risk matters for aggregate pricing, we assume that an individual consumption draw relates to the aggregate consumption draw with the following log-normal heteroskedastic process:

$$\Delta c_{t+1}^{i} = \log \left(\frac{\delta_{t+1}^{i}}{\delta_t^{i}} \frac{C_{t+1}}{C_t} \right) \sim \mathcal{N}(\mu_{c_t^i}, \sigma^2_{i,t}) \tag{39}$$

where the individual consumption exposures satisfy:

$$\int_i \delta_t^i \, di = 1, \quad \log \left(\frac{\delta_{t+1}^{i}}{\delta_t^{i}} \right) \sim \mathcal{N}(\mu_{\delta_t}, \sigma^2_{\delta_t}), \tag{40}$$

$\forall t$. The conditional means and variances for the corresponding normal distributions are noted in the parentheses above.[20] Together, Equations (39) and (40) thus imply $\mu_{c_t^i} = \mu_t^\delta + \mu_{c_t^i}$ and $\sigma_{i,t}^2 = \sigma_{\delta,t}^2 + \sigma_{C,t}^2 + 2\sigma_{\delta_t,C_t}$, where $\sigma_{\delta,t}^2 \equiv var_t\left(\log\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)\right)$, and $\sigma_{\delta_t,C_t} \equiv cov_t\left(\log\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right), \log\left(\frac{C_{t+1}}{C_t}\right)\right)$. Several functional forms for δ_t^i can satisfy our requirements (in particular, δ_t^i sum to one by the law of large numbers). One example, from Constantinides and Duffie (1996) is $\delta_{t+1}^i/\delta_t^i = \exp(z_{t+1}^i x_{t+1} - x_{t+1}^2/2)$ where z_{t+1}^i is distributed as standard normal and is independently distributed from x_{t+1}.[21] We maintain that the Foreign country has a representative agent.

Consider the problem faced by an individual Home household i. No-arbitrage pricing requires that the Euler equations for household i investing in Home bonds and Foreign bonds be satisfied:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-s}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)^{-s}\right] = \frac{1}{R_t}, \tag{41}$$

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \frac{1}{R_t^*} \tag{42}$$

Taking a log expansion and using the SDF definition (38) the Home Euler equation for the Home bond can be written as:

$$-r_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + s^2 var_t(\log(\delta_{t+1}^i)) + 2s^2 cov_t(\log(\delta_{t+1}^i), c_{t+1}))] \tag{43}$$

where the moments of the SDF based on aggregate consumption are given by $\mathbb{E}_t[m_{t+1}] = -s\mu_{C_t}$ and $var_t[m_{t+1}] = s^2\sigma_{C_t}^2$. Moreover, the mean SDF of any two individuals i and j are equated $\mathbb{E}_t[M_{t+1}^i] = \mathbb{E}_t[M_{t+1}^j]$ by risk-sharing. Equation (41) is the analogue of Equation (2). The RHS of Equation (41) has extra terms relative to $var_t(m_{t+1})$ which reflect a precautionary motive from facing idiosyncratic consumption risk.

[20]We repeatedly use the result that if $X \sim \mathcal{N}(\mu_X, \sigma_X)$, $e^X \sim \log\mathcal{N}(\mu_X, \sigma_X)$ and if $Y \sim \mathcal{N}(\mu_Y, \sigma_Y)$, then $Z = XY \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X + \sigma_Y + 2\sigma_{XY})$. From this it follows that $e^Z \sim \log\mathcal{N}(\mu_X + \mu_Y, \sigma_X + \sigma_Y + 2\sigma_{XY})$.

[21]The law of large numbers follows from properties of the normal distribution. Treating x_{t+1} as a constant, and using the moment generating function $M_z(t) = e^{tz}$, $\mathbb{E}[z_{t+1}^i x - x^2/2] = M_z(t)e^{-x^2/2} = e^0 = 1$.

Turning to the Euler for a Home household i investing in Foreign bonds (the analogous equation to equation (5)), and assuming Δe_{t+1} is also jointly normally distributed, we get:

$$-r_t^* = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + s^2 var_t(\log(\delta_{t+1}^i)) + 2s^2 cov_t(\log(\delta_{t+1}^i), c_{t+1})] + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}^i, \Delta e_{t+1}) \quad (44)$$

where,

$$cov_t(m_{t+1}^i, \Delta e_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - s\ cov_t(\log(\delta_{t+1}^i), \Delta e_{t+1})$$

which follows from $m_{t+1}^i = -s\Delta c_{t+1} + \log(\delta_{t+1}^i/\delta_t^i)$ and $m_{t+1} = -s\Delta c_{t+1}$.[22]

This extended economy admits an exchange rate process described by Equation (7).[23] The relevant restrictions imposed on the incomplete markets wedge η_{t+1} differ. Cross-border trade in Home and Foreign bonds imply:

$$\mathbb{E}_t\left[M_{t+1}e^{-\eta_{t+1}}\right] = \mathbb{E}_t\left[M_{t+1}^i\right],$$

$$\mathbb{E}_t\left[M_{t+1}^i \frac{M_{t+1}^*}{M_{t+1}} e^{\eta_{t+1}}\right] = \mathbb{E}_t\left[M_{t+1}^*\right]$$

The resulting restrictions on the incomplete market wedge are given by:

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) + \log \mathbb{E}_t\left[\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)^{-s}\right] - cov_t\left(m_{t+1}^*, -\eta_{t+1} + s\log\left(\delta_{t+1}^i\right)\right), \quad (45)$$

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) - \log \mathbb{E}_t\left[\left(\frac{\delta_{t+1}^i}{\delta_t^i}\right)^{-s}\right] - cov_t\left(m_{t+1}, \eta_{t+1} - s\log\left(\delta_{t+1}^i\right)\right) \quad (46)$$

Relative to the representative agent benchmark, the properties of the exchange rate process change to reflect the idiosyncratic risk distribution in Home.

[22] Lemma 2 provides one specific model (the autarky limit) which connects aggregate shocks with depreciation Δe_{t+1}, giving rise to a covariance between idiosyncratic risk and exchange rate movements. In general, idiosyncratic risk can be related to exchange rates through a number of mechanisms.

[23] This can be seen because (41), (3),(4) and (42) imply:

$$var_t(\Delta e_{t+1}) = cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) + cov_t(\delta_{t+1}^i, \Delta_{t+1})$$

The next proposition details how the model with heterogeneous consumers in the Home economy, with only the Foreign bond internationally traded, can deliver low risk sharing.

Proposition 5 *When only Foreign bonds are internationally traded and there exist a continuum of heterogeneous Home consumers such that Equations (3), (41), and (42) hold, then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if:*

$$\begin{aligned} cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq \quad & var(m^*_{t+1} - m_{t+1}) \\ & - \log \mathbb{E}_t \left[\frac{\delta^i_{t+1}}{\delta^i_t}^{-s} \right] + s\ cov_t(\log(\delta^i_{t+1}), m^*_{t+1} + \eta_{t+1}) \end{aligned}$$

The model can thus deliver low risk-sharing if the non-traded component is sufficiently safe, generalizing Proposition 1 to an environment with idiosyncratic risk. The threshold for this requirement is now endogenously a function of the idiosyncratic risk. Indeed, without relying on the covariance term (i.e. focusing on the case where idiosyncratic and aggregate risk are uncorrelated), the condition is relaxed by the variance of the idiosyncratic consumption risk faced by the agents.[24]

However, the variance term cannot help reconcile low risk-sharing when two risk-free bonds are internationally traded. We now revisit our main result with two internationally traded risk-free bonds.

Proposition 6 *The two-country model with two internationally traded bonds and a continuum of heterogeneous Home consumers characterized by Equations (3), (4), (41), and (42), can deliver $cov_t(m^*_{t+1} - \log(\int_i e^{\Delta c^i_t} di)^{-s}, \Delta e_{t+1}) < 0$ if and only if*

$$1 \geq \rho_{-\delta^i_{t+1}, -\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{s \sigma_t(\log(\delta^i_{t+1}))} \tag{47}$$

[24]Furthermore, we can see that low risk sharing can also be attained when the consumption risk is countercyclical, namely $cov_t(\log(\delta^i_{t+1}), c_t) < 0$. In macroeconomic models (Werning, 2015; Bilbiie, 2018), it has been shown that countercyclical consumption risk offers amplification of economic mechanisms. Auclert et al. (2021) generalize Werning (2015)'s as-if complete markets representative agent result to an open economy setting.

where $\rho_{-\log(\delta^i_{t+1}),-\Delta e_{t+1}} \equiv \frac{cov_t(-s\delta^i_{t+1}, -\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(s\log(\delta^i_{t+1}))}$

Proof. See Appendix A.2. □

In a representative agent economy, exchange rate risk becomes traded (is spanned) when households trade in both Home and Foreign real bonds across borders. Introducing idiosyncratic consumption risk, and allowing this risk to co-move with the exchange rate recovers a non-traded component which can deliver low risk-sharing.[25] If idiosyncratic consumption risk increases with depreciations, Foreign bonds are a poor hedge for domestic consumption and low risk-sharing persists even with trade in Home and Foreign bonds. This mirrors equation (32) from the two SDF model of limited participation. While the mechanisms underlying the two models differ substantially, the conditions under which both models reconcile low risk sharing both rely on the covariance of uninsurable (unspanned) domestic risk with the exchange rate risk.

Note that ex-post heterogeneity alone is enough to obtain low risk sharing. Building on the contributions of Weil (1992) and Krueger and Lustig (2010), consider the special case of a two period environment $t = \{0, 1\}$. Agents are identical at date 0 but face idiosyncratic risk drawn from a common distribution at date 1 $c^i_1 = \log C^i_1 \sim \mathcal{N}(\mu_c, \sigma^i_c)$. Idiosyncratic risk can be correlated to aggregate risk and exchange rate depreciation. As before, it is useful to denote the distribution of individual consumption $c^i_1 = \log(\delta^i_1 C_1) \sim \mathcal{N}(\mu_c, \sigma^2_c + \sigma^2_\delta + 2\sigma_{c\delta})$. Conditions (41) and (42) go through unchanged, therefore Propositions 5 and 6 can be recovered in this environment. In the limit with zero liquidity in Foreign bonds, this two-period model can reflect a no-trade equilibrium where agents consume their endowment in every period. Similarly, one can derive consumption processes from a no-trade equilibrium in the infinite horizon model

[25] Note that we adjust the risk-sharing measure to consider the co-movement of aggregate consumption and exchange rates, in SDF space, consistent with the original definition of the Backus-Smith covariance.

used above, as in Constantinides and Duffie (1996).

3.2.1 How much heterogeneity, revisited

Finally, we do a back of the envelope calculation to ascertain the empirical plausibility of our mechanism in the model with heterogeneous consumers. We start with the estimates of the amount of idiosyncratic risk from US data. Floden and Lindé (2001) estimate the standard deviation of (annual) idiosyncratic earnings risk to 0.9.[26] Acharya, Challe and Dogra (2023) recover an estimate of 0.5 using the evidence in Guvenen, Ozkan and Song (2014). We will consider the range of $[0.5, 0.9]$ for σ_δ. We assign $\sigma_\delta = 0.5$, to construct conservative bounds for condition (47). We set the inter-temporal elasticity of substitution to 0.1 based on the evidence in Best, Cloyne, Ilzetzki and Kleven (2020), which implies $s = 10$. Together, these values imply that to achieve low risk-sharing we require a correlation coefficient that is at least as high as :

$$\rho_{-\delta^i_{t+1}, -\Delta e_{t+1}} \geq \frac{0.33}{5} = 0.066 \tag{48}$$

Estimation for the correlation is more involved. However, we can arrive at an estimate indirectly. Acharya et al. (2023) assume that the idiosyncratic risk process is $\sigma^2_{\delta,t} = \sigma^2_\delta + \phi c_t$, where ϕ is the measure of cyclicality of income risk and they set ϕ to -5.76. Further, assume that the (log) consumption process can be decomposed into orthogonal components $c_t = \alpha \Delta e_t + \nu_t$.[27] Verner and Gyöngyösi (2020) document that an approximately 30% depreciation of Hungarian forint (against the euro) was associated with increase in debt of 10% of disposable income. An average marginal propensity to consume of 0.22 (measured as the percentage points decline in non-durable consumption from a percentage point increase in debt) implies a value of

[26] Auclert, Rognlie and Straub (2023) use Floden and Lindé (2001)'s estimate in their calibration.

[27] Note that in the example process $\delta^i_{t+1}/\delta^i_t = z^i_{t+1}x_{t+1} - x^2_{t+1}/2$, this amounts to assuming x^2_{t+1} is a linear function of aggregate consumption (as in Constantinides and Duffie (1996)) and exchange rates.

$\alpha = -0.073$. Using the functional forms, the implied $cov_t(-\delta^i_{t+1}, -\Delta e_{t+1}) = \alpha \frac{\phi}{2} \sigma^2_t(\Delta e_{t+1}) = 0.073 \times 2.88 \times 0.11 = 0.016$, and hence $\rho_{-\delta^i_{t+1}, -\Delta e_{t+1}} = 0.14$, significantly higher than required.

4. Conclusion

A classical strand of the literature in international macroeconomics has focused on formulating goods-market mechanisms which generate a negative relationship between consumption growth and depreciation– the opposite sign to that implied by the Backus-Smith condition– as long as financial markets are incomplete. We show that any model which achieves this resolution must rely on a non-traded component to relative prices which is "safe" from a domestic investor perspective. However, Lustig and Verdelhan (2019) determine that any two-country model with a representative agent and frictionless trade in Home and Foreign currency denominated risk-free bonds recovers the exchange rate cyclicality implied by complete markets. We show this is because international trade in these two assets make exchange rate movements fully insurable ex-ante, resulting in redistribution which makes ex-post exchange rate movements risky.

We propose two generalizations of the model beyond the representative agent no-arbitrage benchmark– first a model with two SDFs with limited asset market participation and then a model of heterogeneous consumers facing uninsurable idiosyncratic risk. Introducing heterogeneity in SDFs generates a non-traded component to exchange rate movements, and low risk-sharing persists even with cross-border trade in both Home and Foreign bonds. Independent, back-of-the-envelope calculations, based on the presence of arbitrage and micro estimates respectively, suggest both models could plausibly explain observed patterns of risk sharing.

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A. Online Appendix

A.1. Additional Derivations for Section 2.

To find the admissible set of processes, consider the log expansions of the above conditions, assuming joint log normality:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) = -r_{t+1}, \tag{49}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) = -r^*_{t+1} \tag{50}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \tag{51}$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \tag{52}$$

where lower case levels denote logs, e.g. $\log(M_{t+1}) = m_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (49) and (52), and (50) and (51) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(m_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \tag{53}$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(m^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \tag{54}$$

A.2. Proofs to Propositions

Proof to Proposition 1 The Backus-Smith condition is related to the covariance $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ which can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{55}$$

$$= var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{56}$$

Imposing (9) (international trade in the Foreign asset), but not (8) (international trade in the Home asset) as is done in Lustig and Verdelhan (2019), assuming $\mathbb{E}_t[\eta_{t+1}] = 0$, and rearranging yields the result. □

Proof to Corollary 1 The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1}, \eta_{t+1}) - 2cov_t(m_{t+1}, \eta_{t+1})$$

Imposing (8) and (9):

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1})$$

Taking the limit $cov_t(m_{t+1}, \eta_{t+1}) \rightarrow$ (10) would imply $var_t(\Delta e_{t+1}) < 0$ which cannot be an equilibrium. □

Proof to Proposition 2: In Section A.3 below, we show that (12)-(15) imply:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\Delta e_{t+1} - m^*_{t+1} + m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \quad (57)$$

Using (87) and imposing $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \quad (58)$$

In that case, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if $var_t(\Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) < 0$. □

Proof to Corollary 2: Next, suppose we reintroduce trade in risk-free bonds. Then (8) and (9) hold. In particular, introducing a Home internationally trade risk-free bond implies:

$$cov_t(\tilde{r}_{t+1}, \eta_{t+1}) = 0 \quad (59)$$

Introducing a Foreign internationally trade risk-free bond implies:

$$cov_t(\tilde{r}^*_{t+1}, \eta_{t+1}) = 0 \quad (60)$$

□

Proof to Proposition 3 From Section A.4 below:

$$cov_t(m_{t+1}, \eta) = -\frac{s(1-s)}{1-2(1-\phi)} var_t(g_{y_{H,t+1}}), \tag{61}$$

$$cov_t(m^*_{t+1}, \eta) = -\frac{s(1-s)}{1-2(1-\phi)} var_t(g_{y_{F,t+1}}), \tag{62}$$

$$var_t(m_{t+1} - m^*_{t+1}) = s^2 var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}), \tag{63}$$

$$\frac{1}{2} var_t(\eta_{t+1}) = \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) \tag{64}$$

Assuming $var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) = var_t(g_{y_{H,t+1}})$ (i.e no covariance and conditioning on H shocks), under financial autarky, the result is found by computing $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, where Δe_{t+1} is detailed in Lemma 1.

When there is trade in the Foreign bond across borders, (9) implies that $cov_t(m^*_{t+1}, \eta_{t+1}) = -\frac{1}{2} var_t(m_{t+1})$.[28] Imposing this restriction:

$$cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t^{\to FA}(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1})$$

and $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ if:

$$\frac{-s(1-s)}{1-2(1-\phi)} > s^2 - \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 \tag{65}$$

Naturally, $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ at the autarky limit defined by $ToT_t C_{F,t} = C_{H,t}$.

Analogously, with both Home and Foreign bonds internationally traded, (8) also binds. Then,

$$cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t^{\to FA}(m^*_{t+1}, \eta_{t+1}) - cov_t^{\to FA}(m_{t+1}, \eta_{t+1})$$

[28] In general, the two quantities are not equivalent in the FA limit, but do coincide at $\phi\sigma = \frac{1}{2}$, which is outside of the region of interest.

and $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ requires:

$$\frac{1}{2}\left[\frac{1}{1-2(1-\phi)}\right]^2 \leq 0$$

which is only satisfied with equality at $\phi \to \infty$ so that $var_t(\Delta e_{t+1}) = 0$. □

Deriving exchange rate process for section 3.1. We begin by deriving the condition that must be satisfied by an exchange rate progress satisfying no-arbitrage in the generalized model. Combining (2), (3), (26), (30) yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(d_{t+1}, \Delta e_{t+1}) + r^*_{t+1} - r_{t+1} = 0, \quad (66)$$

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) - r^*_{t+1} + r_{t+1} = 0 \quad (67)$$

Combining the above, the restriction that must be satisfied by any exchange rate process which admits no arbitrage is therefore:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(d_{t+1}, \Delta e_{t+1}) \quad (68)$$

Assuming $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \quad (69)$$

Using equations (8) and (31), we can express the covariance term as

$$\begin{aligned} cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) = {} & -\mathbb{E}_t[d_{t+1}] - \frac{1}{2}var_t(d_{t+1}) - cov_t(m^*_{t+1}, d_{t+1}) \\ & -cov_t(d_{t+1}, \eta_{t+1}) - var_t(\eta_{t+1}) \end{aligned} \quad (70)$$

Using equations (30) and (70), we can simplify equation (69) :

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) + \cdots$$

$$\left\{cov_t(m_{t+1}, d_{t+1}) - cov_t(m^*_{t+1}, d_{t+1}) - cov_t(d_{t+1}, \eta_{t+1}) - var_t(\eta_{t+1})\right\},$$

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(d_{t+1}, \Delta e_{t+1}) \tag{71}$$

so equation (68) is satisfied. □

Proof to Proposition 4 The covariance can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}),$$

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \cdots$$

$$\left\{\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m^*_{t+1} + \eta, d_{t+1}) + \mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1})\right\} + \cdots$$

$$\left\{\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1})\right\}$$

Simplifying:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \cdots$$

$$\left\{\log \mathbb{E}_t[D_{t+1}] + cov_t(m^*_{t+1} + \eta, d_{t+1})\right\} - var_t(\eta_{t+1}) \tag{72}$$

where $\log \mathbb{E}_t[D_{t+1}] = \mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1})$. Using equation (31), this can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) + \log \mathbb{E}_t D_{t+1} + cov_t(m_{t+1} - \eta_{t+1}), d^*_{t+1}) \tag{73}$$

The model can reconcile Backus-Smith if and only if:

$$var_t(\Delta e_{t+1}) + \log \mathbb{E}_t[D_{t+1}] + cov_t(m^*_{t+1} + \eta_{t+1}, d_{t+1}) \leq 0 \tag{74}$$

Additionally, using equation (30) we get:

$$cov_t(d_{t+1}, -\Delta e_{t+1}) \geq var_t(\Delta e_{t+1}) \tag{75}$$

Finally, the Cauchy Schwarz identity implies:

$$cov_t(d_{t+1}, -\Delta e_{t+1}) \leq \sqrt{var_t(d_{t+1}) var_t(\Delta e_{t+1})} \tag{76}$$

Combining the inequalities and dividing by $\sigma_t(d_{t+1})$ yields the result. □

Proof to Corollary 4 With heterogeneous marginal investors in the domestic country, when only the Foreign bond is traded across borders, the relevant Euler equations are (2), (3),(26) and (28). Using (56) but replacing (9) by (31) yields the result. □

Proof to Proposition 6. First, we deal with constructing the correct measure of the Backus-Smith correlation, in SDF space, when there are heterogeneous agents. If log quantities are jointly normally distributed it follows that the log of the Backus-Smith covariance in this economy is equal to the covariance of the logs:[29]

$$\log\left(cov_t\left(\int_i e^{\Delta c^i_{t+1}} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(\log \underbrace{\int_i e^{\Delta c^i_{t+1}} di}_{C_{t+1}}, \Delta e_{t+1}\right) \implies$$

$$\log\left(cov_t\left(\int_i M^i_{t+1} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(m_{t+1}, \Delta e_{t+1}\right) \tag{77}$$

Then combining Equations (3), (4), (41), and (42) leads to:

$$var_t(\Delta e_{t+1}) \leq s\ cov_t(-\log(\delta^i_{t+1}), -\Delta e_{t+1}) \tag{78}$$

Using the Cauchy-Schwarz identity as in Proposition 5 gives the result. □

Proof to Lemma 2 The volatility of $\hat{m}_{t+1}$ is given by:

$$var_t(\hat{m}_{t+1}) = var_t(m_{t+1}) + var_t(d_{t+1}) + 2cov_t(m_{t+1}, d_{t+1}) \tag{79}$$

[29] In general,

$$cov_t(X_{t+1}, Y_{t+1}) = \mathbb{E}_t[X_{t+1}]\mathbb{E}_t[Y_{t+1}]\left(e^{cov_t(U,V)} - 1\right)$$

when $U, V \sim \mathcal{N}$.

However, since the investors in the Home country share risk, $cov_t(m_{t+1}, d_{t+1})$ is pinned down by equation (30). The result follows by substituting $var_t(\hat{m}_{t+1}) = Kvar_t(m_{t+1})$ in equation (79) and imposing within country risk-sharing equation (30).

Proof to Lemma 3 Consider:

$$var_t(d_{t+1}) = var(\hat{m}_{t+1}) + var(\hat{m}_{t+1}) - 2cov_t(\hat{m}_{t+1}, m_{t+1}),$$

$$var_t(d_{t+1}) = var(\hat{m}_{t+1}) + var(\hat{m}_{t+1}) - 2\rho_t(\hat{m}_{t+1}, m_{t+1})\sigma_t(\hat{m}_{t+1})\sigma_t(m_{t+1})$$

Then,

$$var_t(d_{t+1}) \leq (K+1)var(m_{t+1}) - 2\rho_{\hat{m}_{t+1},m_{t+1}}\sqrt{K}\sigma_t^2(m_{t+1}), \tag{80}$$

Rearranging yields the result. □

A.3. Trade in Risky Assets

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (12)- (15) hold. Assuming joint log normality, the above Euler equations imply:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + cov_t(m_{t+1}, \tilde{r}_{t+1}) = 0, \tag{81}$$

$$\begin{aligned} \mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \cdots \\ cov_t(m_{t+1}, \tilde{r}^*_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) = 0, \end{aligned} \tag{82}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + cov_t(m^*_{t+1}, \tilde{r}^*_{t+1}) = 0, \tag{83}$$

$$\begin{aligned} \mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots \\ cov_t(m^*_{t+1}, \tilde{r}_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) = 0 \end{aligned} \tag{84}$$

Combining (81) and (82):

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) - \mathbb{E}_t[\tilde{r}_{t+1}] - \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$
$$cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) + cov_t(m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \tag{85}$$

Combining (83) and (84):

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) - \mathbb{E}_t[\tilde{r}^*_{t+1}] - \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \cdots$$
$$cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) - cov_t(m^*_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \tag{86}$$

Together, the above conditions yield:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\Delta e_{t+1} - m^*_{t+1} + m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{87}$$

Assuming the exchange rate process is given by $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$ this condition reduces to:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{88}$$

Imposing the exchange rate process, we can derive restrictions to the incomplete market wedge analogous to equations (8) and (9). Then, doing a log expansion from combining equations (15), (81), and the exchange rate process, we get:

$$cov_t(m_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\tilde{r}_{t+1}, \eta_{t+1}) \tag{89}$$

Additionally, equations (13) and (83) imply:

$$cov_t(m^*_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\tilde{r}^*_{t+1}, \eta_{t+1}) \tag{90}$$

The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) = \cdots$$

$$var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{91}$$

which verifies (87), so the exchange rate process is admissible.

A.4. An equilibrium model

To fix ideas, we present an equilibrium two-country, two-good, endowment model solved under the assumption of financial autarky. This allows us to express SDFs and prices as functions of exogenous variables. Financial autarky is not a restrictive assumption for us since we are interested in the sign of covariances when there is trade in assets, and Euler equations apply even in the ϵ liquidity limit. However, to attain joint normality of SDFs and prices, we further need to assume the limit of full home-bias, as in e.g. Itskhoki and Mukhin (2021).

Starting with the static conditions:

$$C = \left[\alpha^{\frac{1}{\phi}} C_H^{\frac{\phi - 1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_F^{\frac{1}{\phi}} \right]^{\frac{\phi}{\phi - 1}} \tag{92}$$

Relative demand for goods requires:

$$\frac{C_F}{C_H} = \frac{1-\alpha}{\alpha} ToT^{-\phi}, \tag{93}$$

$$\frac{C^*_F}{C^*_H} = \frac{1-\alpha^*}{\alpha^*} ToT^{-\phi}, \tag{94}$$

where τ denotes the terms of trade.Market clearing requires:

$$C_H + C^*_H = Y_H \tag{95}$$

$$C_F + C^*_F = Y_F \tag{96}$$

The real exchange rate is given by:

$$\mathcal{E}_{=}\frac{P^*}{P} \tag{97}$$

and the terms of trade:

$$ToT = \frac{P_F}{P_H} \tag{98}$$

The law of one price holds for each good but not the aggregate basket unless $\alpha = \alpha^*$.

Under financial autarky, $PC = P_H Y_H$ and $P^* C^* = P_F Y_F$. Combining this with relative demand yields:

$$C_H + ToT\ C_F = Y_H, \tag{99}$$

$$C_F = \frac{1-\alpha}{\alpha}\left(\frac{1}{ToT}\right)^{\phi} C_H, \tag{100}$$

$$C_H\left[1 + ToT^{1-\phi}\left(\frac{1-\alpha}{\alpha}\right)\right] = Y_H, \tag{101}$$

$$C_H = Y_H\left[1 + ToT^{1-\phi}\left(\frac{1-\alpha}{\alpha}\right)\right]^{-1} \tag{102}$$

$$C_H = Y_H\left[\frac{\alpha}{\alpha + ToT^{1-\phi}(1-\alpha)}\right] \tag{103}$$

For :

$$C_H^* ToT^{-1} + C_F^* = Y_F, \tag{104}$$

$$C_F^* = \frac{1-\alpha^*}{\alpha^*}\left(\frac{1}{ToT}\right)^{\phi} C_H^*, \tag{105}$$

$$C_H^*\left[ToT^{-1} + ToT^{-\phi}\left(\frac{1-\alpha^*}{\alpha^*}\right)\right] = Y_F, \tag{106}$$

$$C_H^* = Y_F\left[\frac{\alpha^* ToT^{-1} + ToT^{-\phi}1-\alpha^*}{\alpha^*}\right]^{-1}, \tag{107}$$

$$C_H^* = Y_F\left[\frac{\alpha^*}{\alpha^* ToT^{-1} + ToT^{-\phi}(1-\alpha^*)}\right] \tag{108}$$

Balanced trade, and the law of one price, requires $\tau_t C_F = C_{H^*}$ in every period. Using relative demand:

$$ToT_t = \frac{\alpha^* \frac{P_t^*}{p_{H,t}}^{\phi} C_t^*}{(1-\alpha)\frac{P_t}{p_{F,t}}^{\phi} C_t} \tag{109}$$

Using autarky again:

$$ToT_t = \frac{\alpha^* \frac{P_t^*}{p_{H,t}}^{\phi} \frac{p_{F,t}^*}{P_t^*} Y_{F,t}^*}{(1-\alpha)\frac{P_t}{p_{F,t}}^{\phi} \frac{p_{H,t}}{P_t} Y_{H,t}} \tag{110}$$

Imposing $\alpha^* = (1-\alpha)$:

$$ToT_t = \underbrace{\frac{P_t^*}{P_t}^{\phi-1}}_{\mathcal{E}_t^{\phi-1}} \frac{Y_{F,t}^*}{Y_{H,t}} \underbrace{\frac{p_{F,t}}{p_{H,t}}^{1+\phi}}_{\tau_t^{1+\phi}} \tag{111}$$

So:

$$ToT_t^{-\phi} = \mathcal{E}_t^{\phi-1} \frac{Y_{F,t}^*}{Y_{H,t}} \tag{112}$$

Using a first order approximation and $q = (2\alpha - 1)\tau$,

$$\tau = \frac{y_H - y_F}{1 - 2\alpha(1-\phi)}, \tag{113}$$

$$\Delta e = (2\alpha - 1) * \Delta\tau \tag{114}$$

We now show that at the limit of $\alpha \to 1$, the model coincides with a two-country, two-good (no trade), CAPM. Home and Foreign consumption is given by:

$$c_t = c_{H,t} = y_{H,t} \tag{115}$$

$$c_t^* = c_{F_t}^* = y_{F,t} \tag{116}$$

Assuming $g_{y_{H,t}},\ g_{y_{F,t}} \sim \mathcal{N}(\mu, \sigma_y^2)$, then $m_{t+1}^{(*)} \sim \mathcal{N}(-s\mu, s^2\sigma_{y_i}^2)$, $i \in \{H, F\}$. We can then construct:

$$\eta_{t+1} = (g_{y_{H,t+1}} - g_{y_{F,t+1}})\frac{1-s}{1-2\alpha(1-\phi)} \tag{117}$$

using (7).

Next, we show our approximated equilibrium model delivers the Backus-Smith puzzle and its resolution. In particular:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = cov_t\left(-s(g_{y_{F,t+1}} - g_{y_{H,t+1}}), \tfrac{1}{1-2(1-\phi)}(g_{y_{H,t+1}} - g_{y_{F,t+1}}\right) < 0 \quad (118)$$

if $\phi < \frac{1}{2}$ – consistent with Corsetti et al. (2008). However, we are able to go a step further and explain why the mechanism goes through in the one-traded asset case. Notice that with no trade in assets– the coefficient of risk aversion s does not feature.

The figure below illustrates the range of parameters for which Proposition 3 is satisfied.

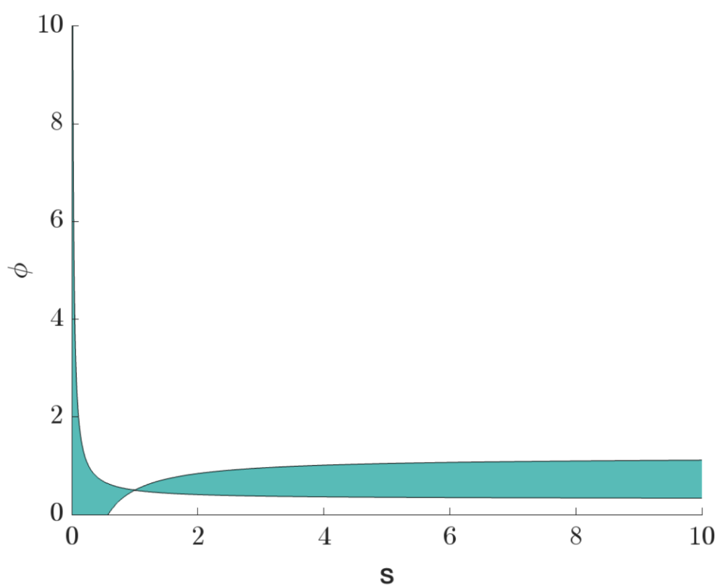

Figure 1: *Proposition 3 (ii)*

Notes: Shaded region reflects parameters for which the model can satisfy the empirical Backus Smith correlation, at the limit of financial autarky and full home-bias.

B. A calibrated two country open economy model

Below, we present a simple version of the endowment economy from Corsetti et al. (2008). Since only one bond is internationally traded, only equations (2) - (4) hold, so we refer to this model as the 3-Euler equation model.

The representative agent derives utility from consumption:

$$u(C_t) = \beta(C_t)\frac{C_t^{1-s}}{1-s} \tag{119}$$

where the consumption bundle is given by:

$$C_t = \left[\alpha^{\frac{1}{\phi}} C_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{120}$$

where ϕ is the trade elasticity. To ensure stationarity we use Uzawa (endogenous) discount factors, see Bodenstein (2011),

$$\beta(C_t) = \omega(C_{t-1})^{-u} \tag{121}$$

Home agents receive an endowment of their domestic good. They also invest in their domestic bonds and "an international bond" which pays in units of Home aggregate consumption and is zero in net supply. The Home agent faces the following budget constraint:

$$P_tC_t - P_{H,t}Y_{H,t} \leq R_tB_{t-1} - B_t + \mathcal{E}_t(R_t^*B_{t-1}^F - B_t^F) \tag{122}$$

Foreign agents face an analogous maximization but purchase only the Foreign bond.

Goods market clearing requires:

$$C_{H,t} + C_{H,t}^* = Y_{H,t}C_{F,t} + C_{F,t}^* = Y_{F,t}^*$$

where $Y_{H,t} = \rho Y_{H,t-1} + (1-\rho)Y_H + \epsilon$, $Y_{F,t}^* = \rho Y_{F,t-1}^* + (1-\rho)Y_F^* + \epsilon_t$. Bond market clearing requires:

$$B_t = 0,$$

$$B_t^* + B_t^F = 0$$

Returning to the financial side of the model, the Home agents' inter-temporal allocation satisfies:

$$\mathbb{E}_t[M_{t+1}R_{t+1}] = 1, \tag{123}$$

$$\mathbb{E}_t\left[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}R^*_{t+1}\right] = 1, \tag{124}$$

whereas Foreign agents face:

$$\mathbb{E}_t[M^*_{t+1}R^*_{t+1}] = 1 \tag{125}$$

The international risk sharing condition in the model is given by:

$$\begin{gathered}\mathbb{E}_t[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}] = \mathbb{E}_t\left[M^*_{t+1}\right] \leftrightarrow \\ \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\right]\end{gathered} \tag{126}$$

Critically, if the Foreign risk-free bond was also traded then the second risk-sharing condition below would also need to be satisfied:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\right] = \mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] \tag{127}$$

Notice that (126) and (127) are the same if approximated to first order, but in general will imply significantly different results.

B.1. Quantitative results

We consider the following calibration: $\omega = 0.96, s = 1, \phi \in \{0.5, 1, 2\}, \alpha = 0.85, \alpha^* = 1 - \alpha, \rho = 0.96, Y_H = 1, u = 0.01$. We also contrast the model to the complete markets case, where (126) is replaced by (1).

Figure 2 below illustrates the Cole-Obstfeld result. The one bond economy perfectly approximates the complete markets allocation for $\phi = 1$. Specifically, $var(\eta_t) = 0$ and $corr_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = 1$.

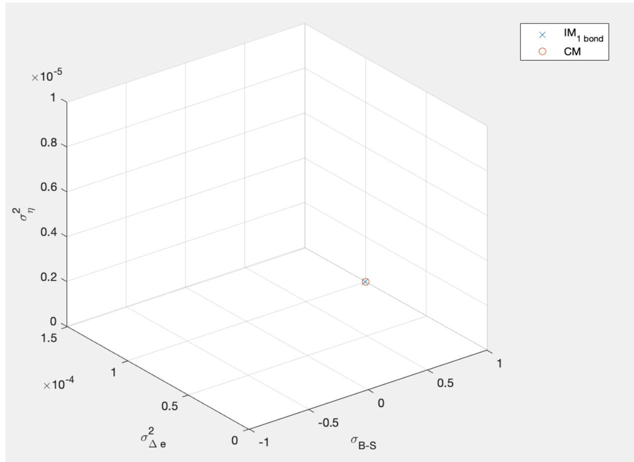

Figure 2: *Cole-Obstfeld parameterization.*

In this instance, financial markets are indeed irrelevant. Figure 3, contrasts the pattern of transmission, the volatility of the exchange rate and, critically, the volatility of non-traded component for $\phi = 0.5$. The Backus-Smith correlation is significantly negative, the volatility of exchange rates rises and the volatility of the IM wedge rises.

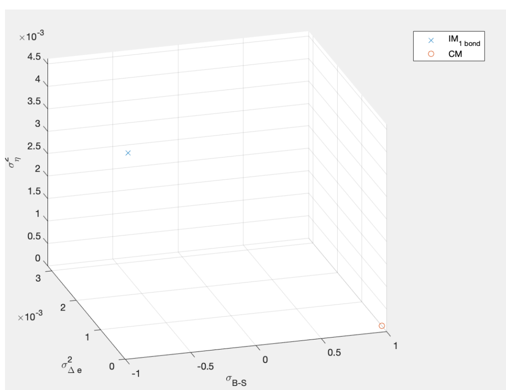

Figure 3: $\phi = 0.5$.

Financial markets here matter – incompleteness allows the model to reconcile the data, but introducing a second internationally traded bond kills the result.

Finally, we evaluate what drives the negative Backus-Smith coefficient in the 3 Euler model, in view of conditions (10). Figure 3 evaluates the various quantities.

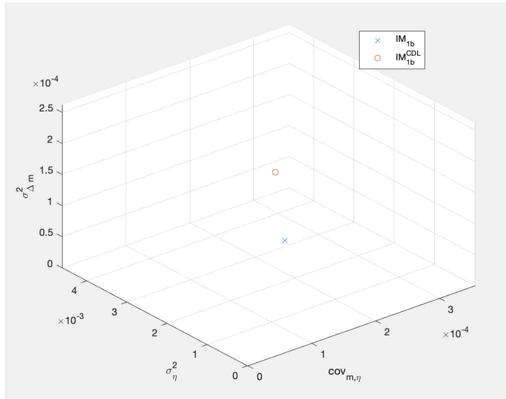

Figure 4: $\phi = 0.5$. *Evaluating Proposition 3.*

Lowering the trade elasticity, raises $cov_t(m_{t+1}, \eta_{t+1}$, lowers $var_t(m^*_{t+1} - m_{t+1})$ and increases $var_t(\eta_{t+1})$, all consistent with condition (10) being violated, so that $\rho^{BS} < 0$. However, the rise in $var_t(\eta_{t+1})$ is order of magnitude larger and therefore drives the result. Consistent with the description of the mechanism in Corsetti et al. (2008), the low trade elasticity prevents an increase in demand for Foreign goods following a Home income shock, therefore Home consumption rises without a fall in the Home price – increasing the volatility of the incomplete markets wedge (or non traded risk).

C. Example for Heterogeneous Marginal Investors.

To gain concrete understanding of condition (32), we flesh out the financial market structure in the Home economy. The simplest model of heterogeneity consistent with our framework is one where the investor characterized by m_{t+1} and the investor characterized by $\hat{m}_{t+1}$ are identical except the latter participates in financial markets for Foreign assets. Imposing consumption utility structure on the SDFs, $\hat{m}_{t+1} = \log(u'(y_{t+1} + w^H_{t+1} + w^F_{t+1}))$ and $m_{t+1} = \log(u'(y_{t+1} + w^H_{t+1}))$, where y_{t+1} is the value of the Home country's endowment, w^H_{t+1} is wealth after trade in a set of basis assets (e.g. just the Home bond) and w^F_{t+1} is defined as the residual portfolio wealth after trade in both the set of basis assets and the Foreign bond.[30] Assuming for exposition that m^*_{t+1} does not vary a lot and utility is exponential, equation (1) implies:[31]

$$cov_t(-aw^F_{t+1}, \Delta e_{t+1}) \leq 0 \tag{128}$$

In other words, the marginal investor purchases sufficient insurance ex-ante, that the exchange rate is risky ex-post consistent with redistribution– but this investor does not pass the insurance on to the domestic household through domestic asset markets. It is useful to note that the implied comovement of $\hat{m}_{t+1}$ and m_{t+1} in this framework is given by $\alpha^2 var_t(y_{t+1} + w^H_{t+1}) + \alpha^2 cov_t(y_{t+1} + w^H_{t+1}, w^F_{t+1})$, which will depend on how portfolios are formed and the underlying structure of shocks which we have not specified.[32]

[30] w^H is the return on the basis asset portfolio which are freely traded by both investors. Note that the autarky limit is where $cov_t(m_{t+1}, -\Delta e_{t+1}) < 0$, requiring $cov_t(y_{t+1}, -\Delta e_{t+1}) > 0$, consistent with Proposition 2. Moreover, at the autarky limit $m_{t+1} \to \hat{m}_{t+1}$.

[31] As is standard in portfolio choice, exponential utility (CARA) allows us to break the individual components by abstracting from wealth effects. Specifically, $u(C) = -e^{-\alpha C}$.

[32] Corsetti, Dedola and Leduc (2014) discipline portfolios using the data and show there is low-risk sharing when there is trade in one international nominal risk-free bond, and trade in international equities.